Dynamic Programming

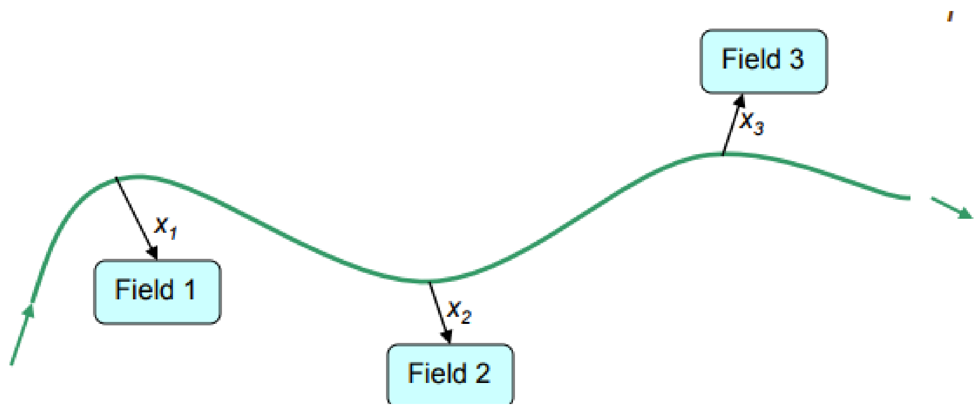
Dynamic programming provides an extremely powerful approach for solving the optimization problems that occur in the operation and planning of **water resource** systems. Dynamic programming provides an extremely powerful approach for solving the optimization problems that occur in the operation and planning of water resource systems. However, its applicability has been somewhat limited because of the large computational requirements of the standard algorithm.

Dynamic Programming : A Sequential or multistage decision making process .

Water Allocation problem is solved as a sequential process using **dynamic** approach.

programming Objectives :-(1) To discuss the Water Allocation Problem (2) To explain and develop **recursive equations** for backward approach (3) To explain and develop recursive equations for forward approach

Consider a canal supplying water for three different crops . Maximum capacity of the canal is Q units of water. Amount of water allocated to each field as xi.



Net benefits (NB) from producing the crops can be expressed as a function of the water allotted.

$$NB_1(x_1) = 5x_1 - 0.5x_1^{\,2}$$
$$NB_2(x_2) = 8x_2 - 1.5x_2^{\,2}$$
$$NB_3(x_3) = 7x_3 - x_3^{\,2}$$

Optimization Problem:

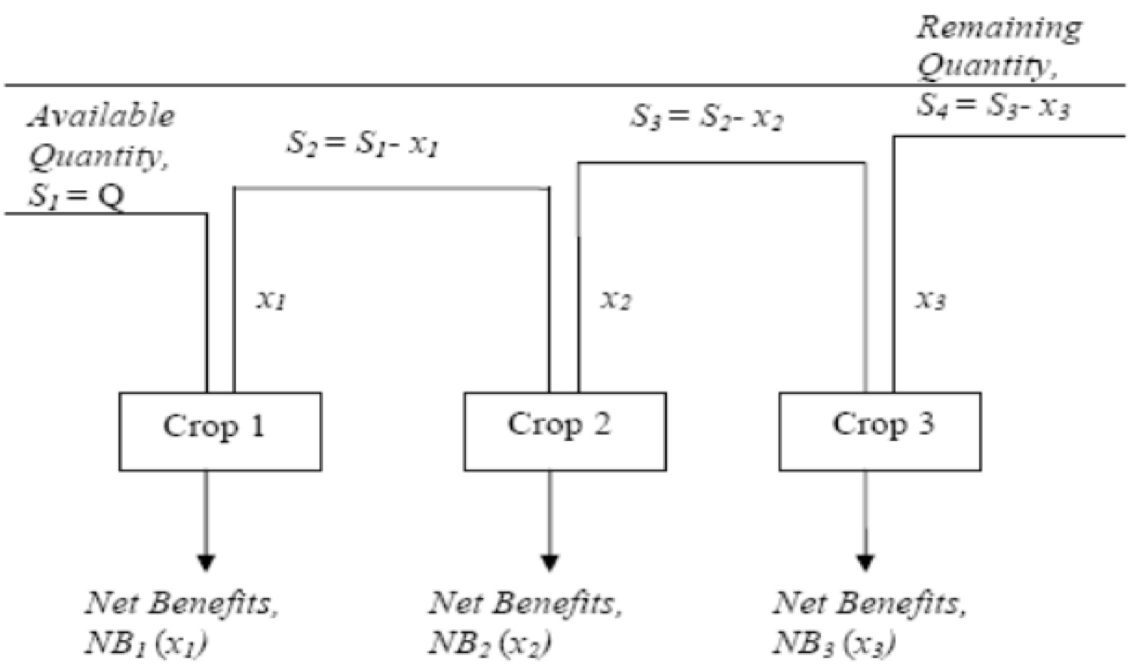
Determine the optimal allocations xi to each crop that maximizes the total net benefits from all the three crops

Structure the problem as a sequential allocation process or a multistage decision making procedure. Allocation to each crop is considered as a decision stage in a sequence of decisions. Amount of water allocated to crop i is xi . Net benefit from this allocation is NBi(xi) Introduce one state variable Si :- Amount of water available to the remaining (3-i) crops . State transformation equation can be written as

$$S_{i+1} = S_i - x_i$$

Sequential Allocation Process

The allocation problem is shown as a sequential process



Backward Recursive Equations Objective function:-- To maximize the net benefits

$$\max \sum_{i=1}^{3} NB_i(x_i)$$

Subjected to the constraints

$$x_1 + x_2 + x_3 \le Q$$
$$0 \le x_i \le Q \qquad for\ i = 1,2,3$$

Let $f_1(Q)$ be the maximum net benefits that can be obtained from allocating water to crops 1,2 and 3

$$f_1(Q) = \max_{\substack{x_1+x_2+x_3 \le Q \\ x_1, x_2, x_3 \ge 0}} \left[\sum_{i=1}^{3} NB_i(x_i) \right]$$

Transforming this into three problems each having only one decision variable

$$f_1(Q) = \max_{\substack{x_1 \\ 0 \le x_1 \le Q}} \left[NB_1(x_1) + \max_{\substack{x_2 \\ 0 \le x_2 \le Q - x_1 = S_2}} \left\{ NB_2(x_2) + \max_{\substack{x_3 \\ 0 \le x_3 \le S_2 - x_2 = S_3}} NB_3(x_3) \right\} \right]$$

Now starting from the last stage, let $f_3(S_3)$ be the maximum net benefits from crop 3.

State variable S_3 for this stage can vary from 0 to Q

Thus,

$$f_3(S_3) = \max_{\substack{x_3 \\ 0 \le x_3 \le S_3}} NB_3(x_3)$$

But $S_3 = S_2 - x_2$. Therefore $f_3(S_3) = f_3(S_2 - x_2)$

Hence,

$$f_1(Q) = \max_{\substack{x_1 \\ 0 \le x_1 \le Q}} \left[NB_1(x_1) + \max_{\substack{x_2 \\ 0 \le x_2 \le Q - x_1 = S_2}} \left\{ NB_2(x_2) + f_3(S_2 - x_2) \right\} \right]$$

Now, let $f_2(S_2)$ be the maximum benefits derived from crops 2 and 3 for a given quantity S_2 which can vary between 0 and Q

Therefore,

$$f_2(S_2) = \max_{\substack{x_2 \\ 0 \le x_2 \le Q - x_1 = S_2}} \left\{ NB_2(x_2) + f_3(S_2 - x_2) \right\}$$

Now since $S_2 = Q - x_1$, $f_1(Q)$ can be rewritten as

$$f_1(Q) = \max_{\substack{x_1 \\ 0 \le x_1 \le Q}} \left[NB_1(x_1) + f_2(Q - x_1) \right]$$

Once the value of $f_3(S_3)$ is calculated, the value of $f_2(S_2)$ can be determined, from which $f_1(Q)$ can be determined.

Let the function $f_i(S_i)$ be the total net benefit from crops *1* to *i* for a given input of S_i which is allocated to those crops.

Considering the first stage,

$$f_1(S_1) = \max_{\substack{x_1 \\ x_1 \le S_1}} NB_1(x_1)$$

Solve this equation for a range of S_1 values from *0* to *Q*

Considering the first two crops, for an available quantity of S_2, $f_2(S_2)$ can be written as

$$f_2(S_2) = \max_{\substack{x_2 \\ x_2 \le S_2}} \left[NB_2(x_2) + f_1(S_2 - x_2) \right]$$

S_2 ranges from *0* to **Q**

Considering the whole system, $f_3(S_3)$ can be expressed as,

$$f_3(S_3) = \max_{\substack{x_3 \\ x_3 \le S_3 = Q}} \left[NB_3(x_3) + f_2(S_3 - x_3)\right]$$

If the whole **Q** units of water should be allocated then the value of S_3 can be taken as equal to **Q**

Otherwise, S_3 will take a range of values from *0* to **Q**

The basic equations for the water allocation problem using both the approaches are discussed

Consider the example previously discussed with the maximum capacity of the canal as **4 units**. The net benefits from producing the crops for each field are given by the functions below.

$$NB_1(x_1) = 5x_1 - 0.5x_1^2$$
$$NB_2(x_2) = 8x_2 - 1.5x_2^2$$
$$NB_3(x_3) = 7x_3 - x_3^2$$

The possible net benefits from each crop are calculated according to the functions given and are given in Table 1.

Table 1

x_i	$NB_1(x_1)$	$NB_2(x_2)$	$NB_3(x_3)$
0	0.0	0.0	0.0
1	4.5	6.5	6.0
2	8.0	10.0	10.0
3	10.5	10.5	12.0
4	12.0	8.0	12.0

$f_3(S_3) = \max\ NB_3(x_3)$ with the range of S_3 from 0 to 4.

The calculations for this stage are shown in the table 2.

Table 2

State	$NB_3(x_3)$						$f_3(S_3)$	x_3^*
S_3	x_3:	0	1	2	3	4		
0		0					0	0
1		0	6				6	1
2		0	6	10			10	2
3		0	6	10	12		12	3
4		0	6	10	12	12	12	3,4

Next, by considering last two stages together, the suboptimization function is

$$f_2(S_2) = \max\ [NB_2(x_2) + f_1(S_2 - x_2)]$$

This is solved for a range of S_2 values from 0 to 4. The value of $f_3(S_2 - x_2)$ is noted from the previous table. The calculations are shown in Table 3.

Table 3

State S_2	x_2	$NB_2(x_2)$	(S_2-x_2)	$f_3(S_2-x_2)$	$f_2(S_2)=$ $NB_2(x_2)+$ $f_3(S_2-x_2)$	$f_2^*(S_2)$	x_2^*
0	0	0	0	0	0	0	0
1	0	0	1	6	6	6.5	1
	1	6.5	0	0	6.5		
2	0	0	2	10	10	12.5	1
	1	6.5	1	6	12.5		
	2	10	0	0	10		
3	0	0	3	12	12	16.5	1
	1	6.5	2	10	16.5		
	2	10	1	6	16		
	3	10.5	0	0	10.5		
4	0	0	4	12	12	20	2
	1	6.5	3	12	18.5		
	2	10	2	10	20		
	3	10.5	1	6	16.5		
	4	8	0	0	8		

Finally, by considering all the three stages together, the sub-optimization function is

$f_1(Q)=\max\limits_{\substack{x_1\\0\le x_1\le Q}}\left[NB_1(x_1)+f_2(Q-x_1)\right]$. The value of $S_1=Q=4$. The calculations are shown in the table 4.

Table 4

State $S_1 = Q$	x_1	$NB_1(x_1)$	$(Q-x_1)$	$f_2(Q-x_1)$	$f_1(S_1) = NB_1(x_1) + f_2(Q-x_1)$	$f_1^*(S_1)$	x_1^*
4	0	0	4	20	20	21	1
	1	4.5	3	16.5	21		
	2	8	2	12.5	20.5		
	3	10.5	1	6.5	17		
	4	12	0	0	12		

Now, backtracking through each table to find the optimal values of decision variables, the optimal allocation for crop 1, $x_1^* = 1$ for a S_1 value of 4. This will give the value of S_2 as $S_2 = S_1 - x_1 = 3$. From Table 3, the optimal allocation for crop 2, x_2 for $S_2 = 3$ is 1. Again, $S_3 = S_2 - x_2 = 2$. Thus, x_3* from Table 2 is 2. The maximum total net benefit from all the crops is 21. The optimal solution is given below.

$$f^* = 21$$
$$x_1^* = 1$$
$$x_2^* = 1$$
$$x_3^* = 2$$

$$f_1(Q) = \max [NB_1(x_1) + f_2(Q - x_1)] \quad \text{----------------1}$$

$$f_2(S_2) = \max[NB_2(x_2) + f_1(S_2 - x_2)] \quad \text{----------------2}$$

$$f_3(S_3) = \max\ NB_3(x_3) \quad \text{--3}$$

Equations (1), (2) and (3) are called as the recursive set of Equations.

Tutorial Class problem

(Q) Consider that a quantity of water=Q has to be allotted to three users denoted by j-1,2,3 where (j) is the user. The problem is to supply three water quantities X_1 ,X_2 and X_3 to three users 1,2 and 3 so as to maximize the total net benefits.